Signed Graceful Graph

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Abstract

Graceful labelling was introduced by Rosa in 1967[4] . A graph which admits a graceful labelling is said to be a graceful graph. For signed graph S=(V, E) with m positive edges and n negative edges let f:V(S)⟶{0,1,2..,m+n} be such that in the induced edge function defined by $g_f = s(uv)\,|f(u) - f(v)|$ for all $uv \in E(G)$, the set of numbers received on the positive edges of S is {1,2,..,m} and the set of numbers received on the negative edges of S is {-1,-2,..,-n} respectively. If S admits such an encoder then S is called graceful signed graph[1] . In this paper, a new concept namely 'Signed graceful graph' is introduced. A graph is said to be signed gracefully if it can be given a signed structure which gives rise to a graceful signed graph. A graph which can be signed gracefully is called a signed graceful graph. Signed graceful cycles and one point union of two cycles are studied in this paper.

Subject Classification: 05C78

Key words: Graceful graph, Graceful signed graph, Signed graceful graph.

1 Introduction

Graph labellings play a vital role in communication networks. Starting from β-valuations introduced by Rosa in 1962[4], thousands of labellings have been defined according to the need of practical problems. Among these the most celebrated one is graceful labelling. This concept has been extended to 'Signed graphs', a strong graph theory tool used in Socialogy. Mukti Acharya and T.singh call this as 'Graceful signed labelling'[1]. Labelling families of graphs gracefully is a routine problem whereas proving the non existence of such a labelling for a graph is a challenging one. It is due to the scarcity of necessary and sufficient condition for a graph to be graceful.

In this paper a new concept namely 'Signed graceful graphs' has been introduced .An attempt has been made to extend the available condition for an Eulerian graph to be graceful to signed graphs whose underlying graphs are Eulerian. The initial but important families of graphs namely cycles and one point union of cycles are studied.

2 Preliminaries

Definition: 2.1 *A graceful labelling of a graph G with q edges is an injection* $f: V(G) \longrightarrow \{0,1,2,\ldots.q\}$ *such that when each edge* $xy \in E(G)$ *is assigned the label* $|f(x) - f(y)|$ *all of the edge labels are distinct.A graph which admits a graceful labelling is said to be a graceful graph.*

Definition: 2.2 *[4] Let S=(V,E) be a signed graph with m positive edges and n negative edges.If* $f: V(S) \longrightarrow \{0,1,2,\ldots..,m+n\}$ *be such that in the induced edge function defined by* $g_f = s(uv)\,|f(u) - f(v)|$ *for all* $uv \in E(S)$*, the set of numbers received on the positive edges of S is* $\{1,2,,3,\ldots..,m\}$ *and the set of numbers received on the negative edges of S is* $\{-1,-2,-3,\ldots.,-n\}$ *respectively. If S admits such an encoder then S is called graceful signed graph.*

Note 2.3 *For further discussion instead of* $\{-1,-2,-3,\ldots.,-m\}$ *let us use labels for negative edges* $\{1,2,3,\ldots.,m\}$ *itself.*

Theorem: 2.4 *[5] Let us suppose that integers , not necessarily distinct are attributed to the vertices of a graph G, and that each edge of G is given an edge label equal to the absolute* *difference of the corresponding vertex values.Then the sum of the edge labels around any circuit of G is even.*

Theorem: 2.5 *[5] Let G be an Eulerian graph of size q. A necessary condition for G to be graceful is that* $\left[\frac{q+1}{2}\right]$ *be even.That is, q ≡1(mod 4),2(mod 4), then G cann't be graceful. In fact, G cann't be binary labeled.*

3 Signed Graceful Graph

Definition: 3.1 *A graph is said to be signed gracefully if it can be given a signed structure which gives rise to a graceful signed graph.*

Example: 3.2 *Consider the cycle* C_6*.*

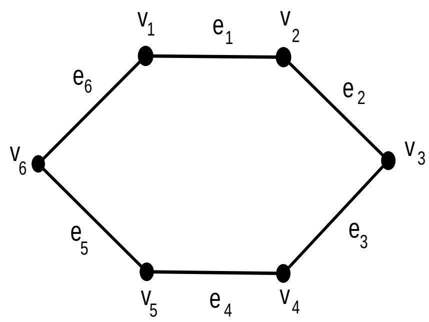


Figure 1.The cycle C_6

In C_6*,Number of edges=6, Since* $6 \equiv 2(mod\ 4)$

By theorem 2.5,C_6 *is not a graceful graph*

Assign positive sign for 5 edges and negative sign for one edges

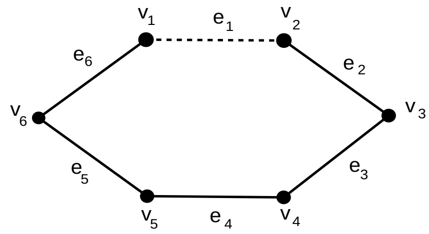


Figure 2. $\mathrm{C}_n(5,1)$

Let V(G)=$\{v_1, v_2, v_3, v_4, v_5, v_6\}$

Define f:V(G)$\longrightarrow$*{0,1,2,3,4} by* $f(v_1) = 0, f(v_2) = 1, f(v_3) = 3, f(v_4) = 2.$

Edge labels are given by $f^*(e_1) = 1, f^*(e_2) = 5, f^*(e_3) = 3, f^*(e_4) = 1, f^*(e_5) = 2, f^*(e_6) = 4.$

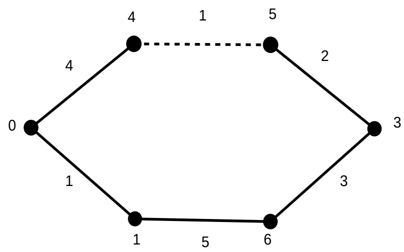


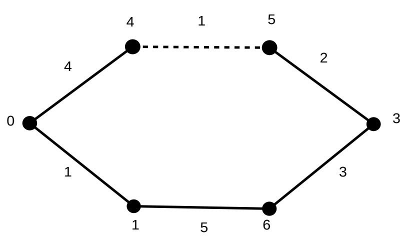
Figure 3.Graceful signed labelling of C_6

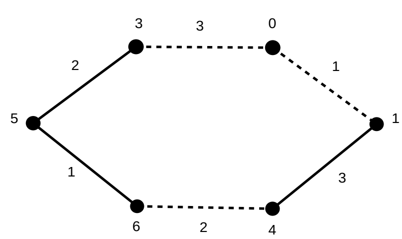
Therefore, C_6 *is a signed graceful graph.*

Note 3.3 *Any graceful graph is a signed graceful graph.*

Note 3.4 *There can be more than one signed structure for G, making it a graceful signed graph.*

Illustration





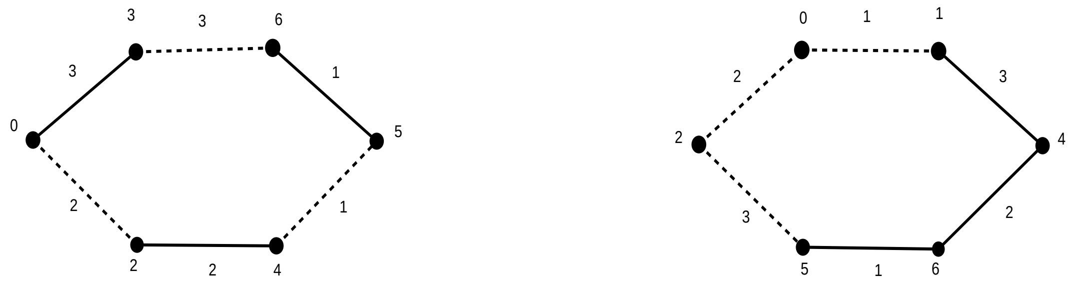


Figure 4. Signed Structure for C_6 making it graceful

Theorem: 3.5 *Let S be a signed graph with m positive edges and n negative edges,whose underlying graph is Eulerian. If S is a graceful signed graph then* $\left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]$ *should be even.*

proof:

If S is a graceful signed graph ,then there exist a graceful signed labelling under which the edges receive labels $\{1,2,3,....,m,1,2,3,....n\}$.

Since underlying graph of S is Eulerian, these labels occur on a circuit.

By theorem 2.4, sum of there edge labels should be even.

Hence sum of the odd edge labels should be even.

It is possible only when total number of odd labeled edges is even.

Among the m positive edges $\left[\frac{m+1}{2}\right]$ *will be odd labeled and among the n negative edges* $\left[\frac{n+1}{2}\right]$ *will be labeled.*

Therefore, $\left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]$ *should be even.*

Theorem: 3.6 C_{4m+1} *is not a signed graceful graph.*

proof:

Given a cycle C_{4m+1} *with 4m+1 edges which is odd.*

C_{4m+1} *is* Eulerian.

All possible ways of signed graceful labellings on C_{4m+1} *are tabulated below*

Labels on negative edges	*Labels on positive edges*	*Total number of odd labeled edges*
1	$1, 2, ..., 4m$	$2m + 1$
$1, 2$	$1, 2, ..., 4m$	$2m + 1$
.	.	.
.	.	.
.	.	.
$1, 2, ..., k$	$1, 2, ..., 4m + 1 - k$	$\left[\frac{k+1}{2}\right] + \left[\frac{4m+2-k}{2}\right]$

Table 1.

Here,two cases arise

Case(i) k is odd

$$\text{Total number of odd labeled edges} = \left[\frac{k+1}{2}\right] + \left[\frac{4m+2-k}{2}\right]$$
$$= \frac{k+1}{2} + \frac{4m+1-k}{2}$$
$$= \frac{4m+2}{2}$$
$$= 2m+1$$

Case(ii) k is even

$$\text{Total number of odd labeled edges} = \left[\frac{k+1}{2}\right] + \left[\frac{4m+2-k}{2}\right]$$
$$= \frac{k}{2} + \frac{4m+2-k}{2}$$
$$= \frac{4m+2}{2}$$
$$= 2m+1$$

In both cases, total number of odd labeled edges are odd.

By theorem 3.5, C_{4m+1} *is not a signed graceful graph.*

Theorem: 3.7 *No signed structure with even number of negative edges on* C_{4m+2} *makes it graceful signed.*

proof:

Cycle C_{4m+2} *has* $4m+2$ *edges*

All possible ways of signed graceful labellings on C_{4m+2} *are tabulated below.*

Labels on negative edges	*Labels on positive edges*	*Total number of odd labeled edges*
1	$1, 2, ..., 4m + 1$	$2m + 2$
$1, 2$	$1, 2, ..., 4m$	$2m + 1$
.	.	.
.	.	.
.	.	.
$1, 2, ..., k$	$1, 2, ..., 4m + 2 - k$	$\left[\frac{k+1}{2}\right] + \left[\frac{4m+2-k}{2}\right]$

Table 2.

If k is even

Total number of odd labeled edges$=\left[\frac{k+1}{2}\right]+\left[\frac{4m+2-k}{2}\right]$

$$=\frac{k}{2}+\frac{4m+2-k}{2}$$

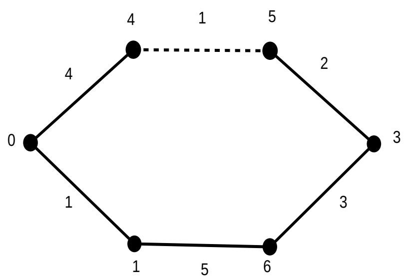
$$=\frac{4m+2}{2}$$

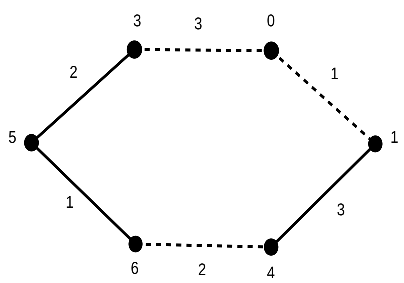
$$=2m+1$$

By theorem 3.5, C_{4m+2} *with a signed structure with even number of negative edges are not a signed graceful graph.*

The graph admits graceful signed labelling only when number of negative edges is odd.

When m=1,Graceful labelling on signed structure of $C_6 \equiv C_{(4*1)+2}$





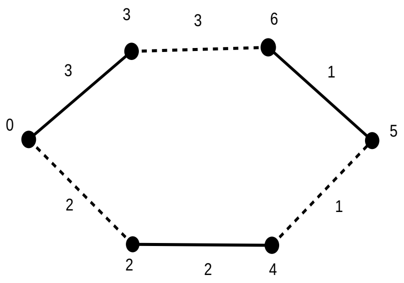


Figure 5. Graceful labelling on signed structure of C_6

When m=2,Graceful labelling on signed structure of $C_{10} \equiv C_{(4*2)+2}$

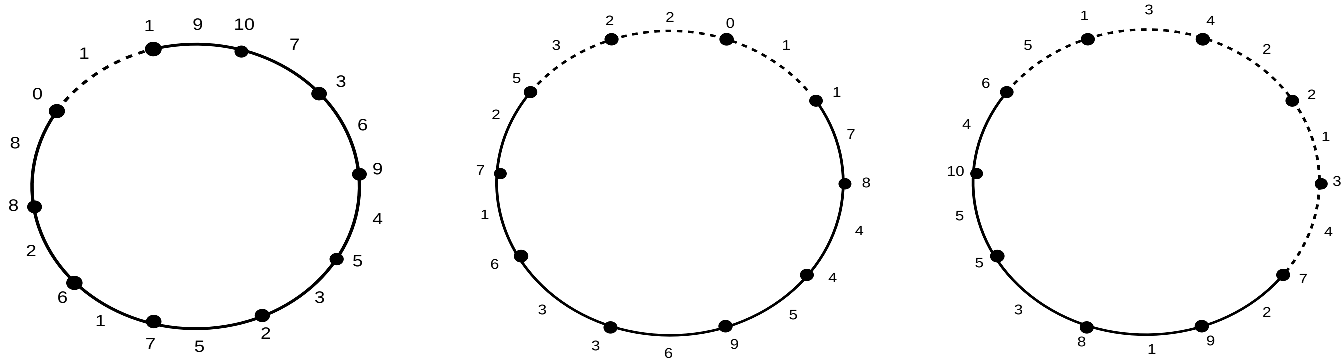
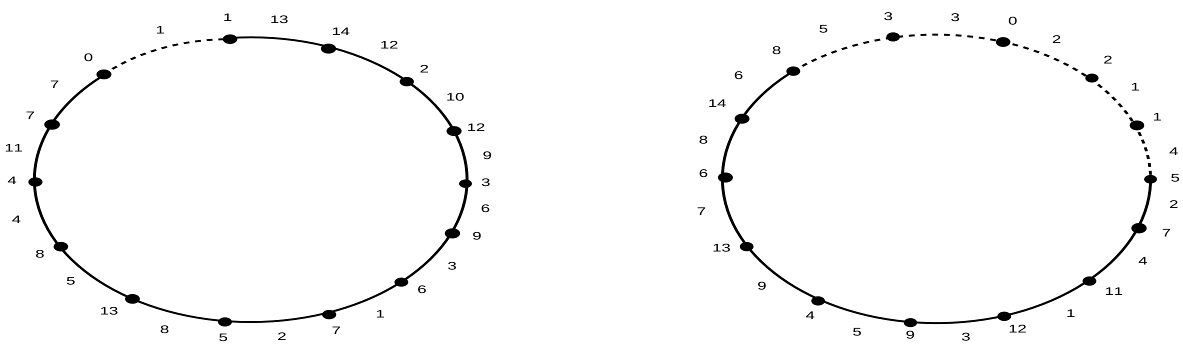


Figure 6.Graceful labelling on signed structure C_{10}

When m=3,Graceful labelling on signed structure of $C_{14} \equiv C_{(4*3)+2}$



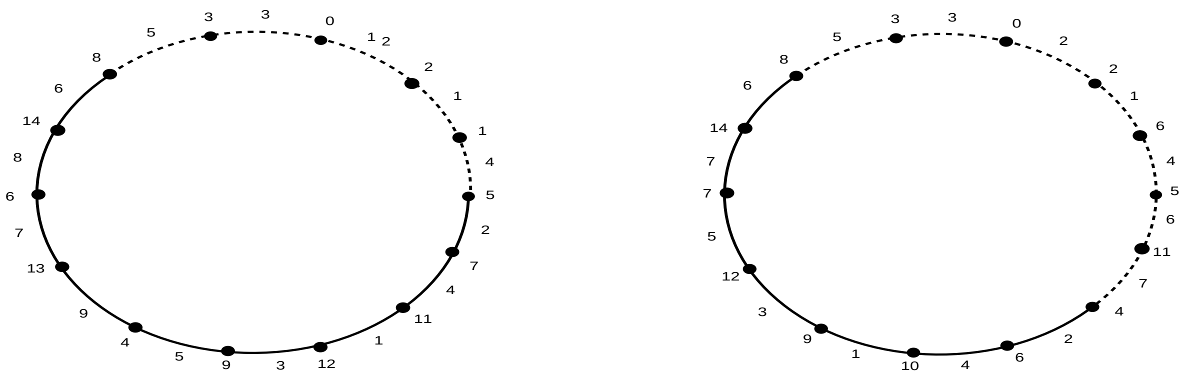


Figure 7. Graceful labelling on signed structure C_{14}

Definition: 3.8 *A one point union of two cycles of equal length say T_{2r+1} is a graph whose vertex set is V(T_{2r+1}) $= \{w_0, u_2, ...u_n, v_2, v_3, ...v_n\}$ and edge set is E(T_{2r+1}) $= \{\{w_0u_2, wu_n, w_0v_1, w_0v_n\} \cup \{u_iu_{i+1}, v_iv_{i+1}/i - 2, 3, ..., n-1\}$ which w_0 has degree 4 and other vertex have degree 2.*

Example: 3.9

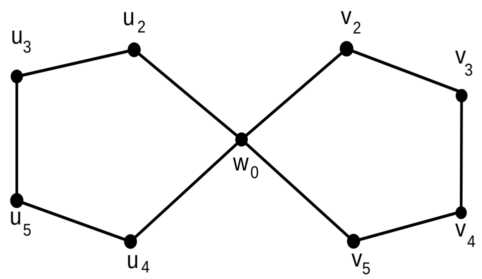


Figure 8. One point union of two cycles C_5

Theorem: 3.10 *No signed structure with even number of negative edges on T_{2r+1} makes it a graceful signed.*

proof:

One point union of two cycles T_{2r+1} has 2r+1 edges.

All possible ways of signed graceful labellings on T_{2r+1} *are tabulated below.*

Labels on negative edges	*Labels on positive edges*	*Total number of odd labeled edges*
1	$1, 2, ..., 4r + 1$	$2r + 2$
$1, 2$	$1, 2, ..., 4r$	$2r + 1$
.	.	.
.	.	.
.	.	.
$1, 2, ..., k$	$1, 2, ..., 4r + 2 - k$	$\left[\frac{k+1}{2}\right] + \left[\frac{4r+3-k}{2}\right]$

Table 3.

If k is even

Total number of odd labeled edges$=\left[\frac{k+1}{2}\right]+\left[\frac{4m+3-k}{3}\right]$

$=\frac{k}{2}+\frac{4r+2-k}{2}$

$=\frac{4r+2}{2}$

$=2r+1$

By theorem 3.5, T_{2r+1} *with a signed structure with even number of negative edges is not a signed graceful graph.*

The graph admits graceful signed labelling only when number of negative edges is odd

When r=1,Graceful labelling on signed structure of $T_3 \equiv T_{(2*1)+1}$

Figure 9.Graceful labelling on signed structure of T_3

When r=2, Graceful labelling on signed structure of $T_5 \equiv T_{(2*2)+1}$

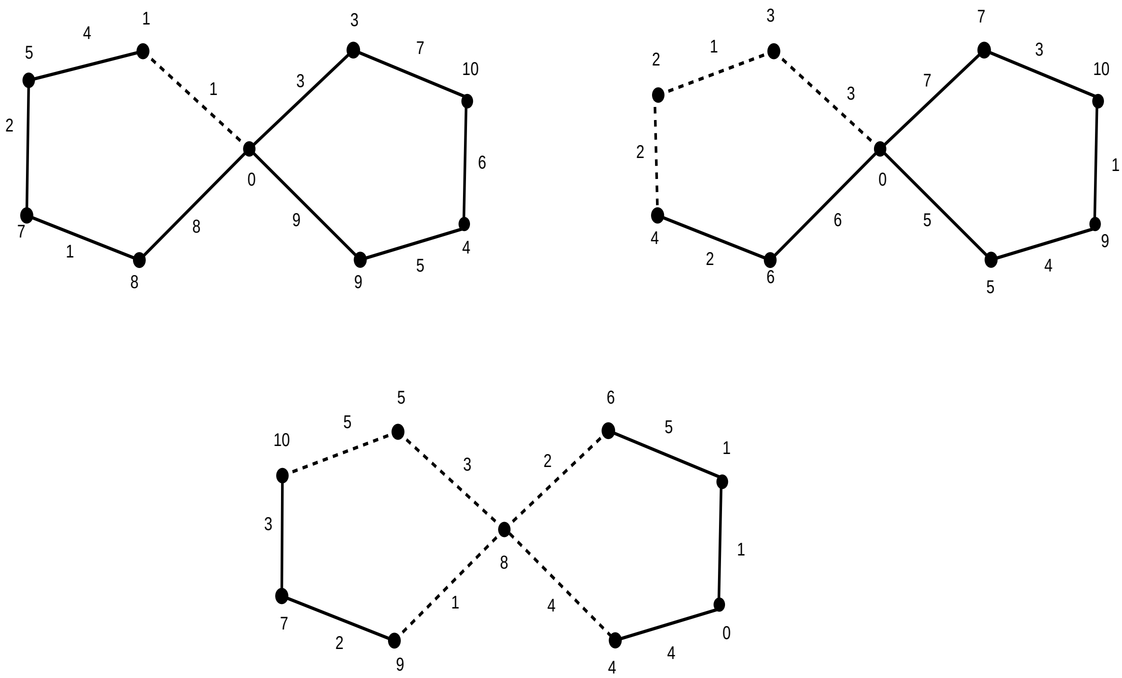
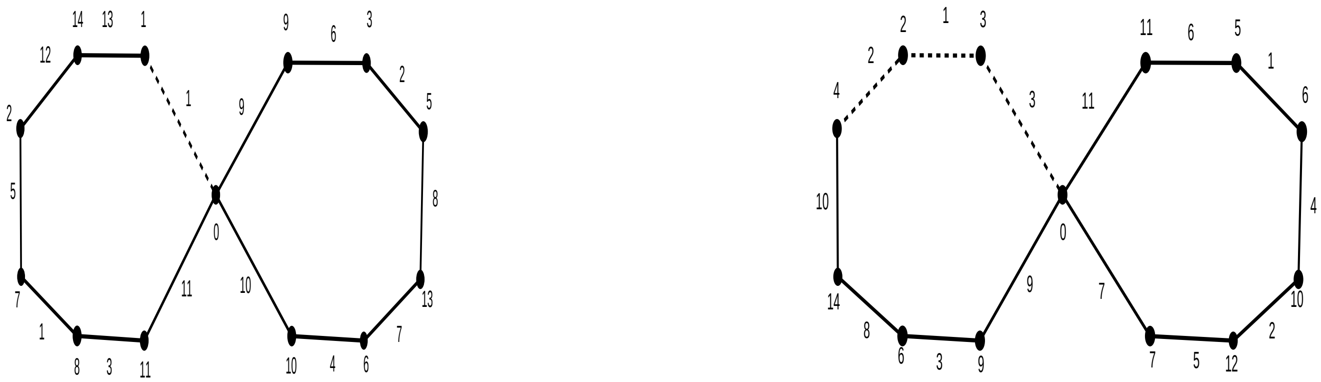


Figure 10.Graceful labelling on signed structure of T_5

When r=3, Graceful labelling on signed structure of $T_7 \equiv T_{(2*3)+1}$



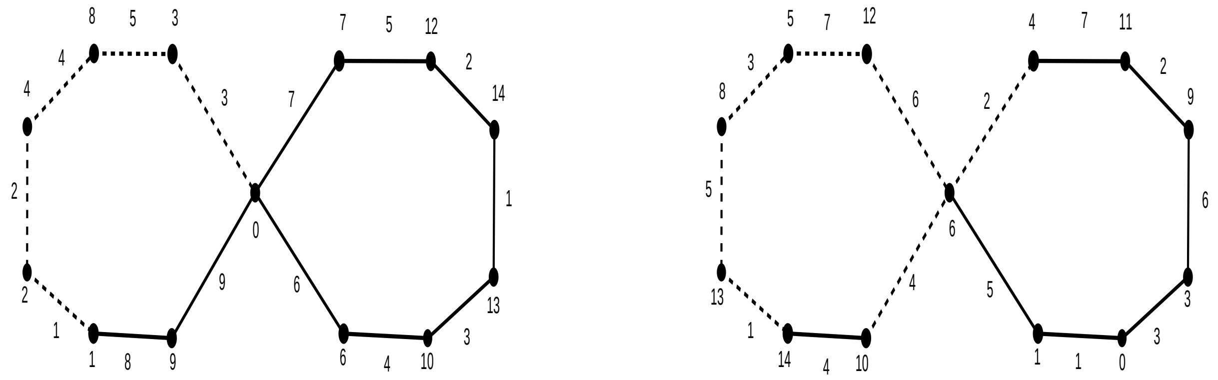


Figure 11. Graceful labelling on signed structure of T_7

The graph admits graceful signed labelling only when number of negative edges is even.
Graceful labelling on signed structure of T_4

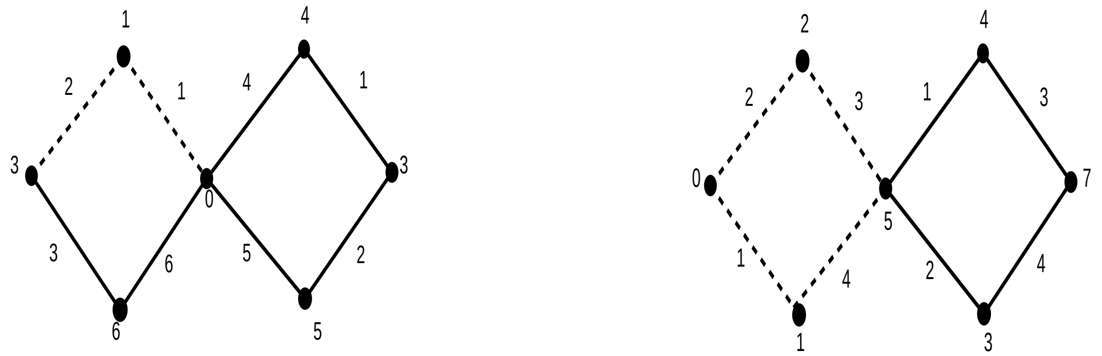


Figure 12.Graceful labelling on signed structure of T_2

Graceful labelling on signed structure of T_6

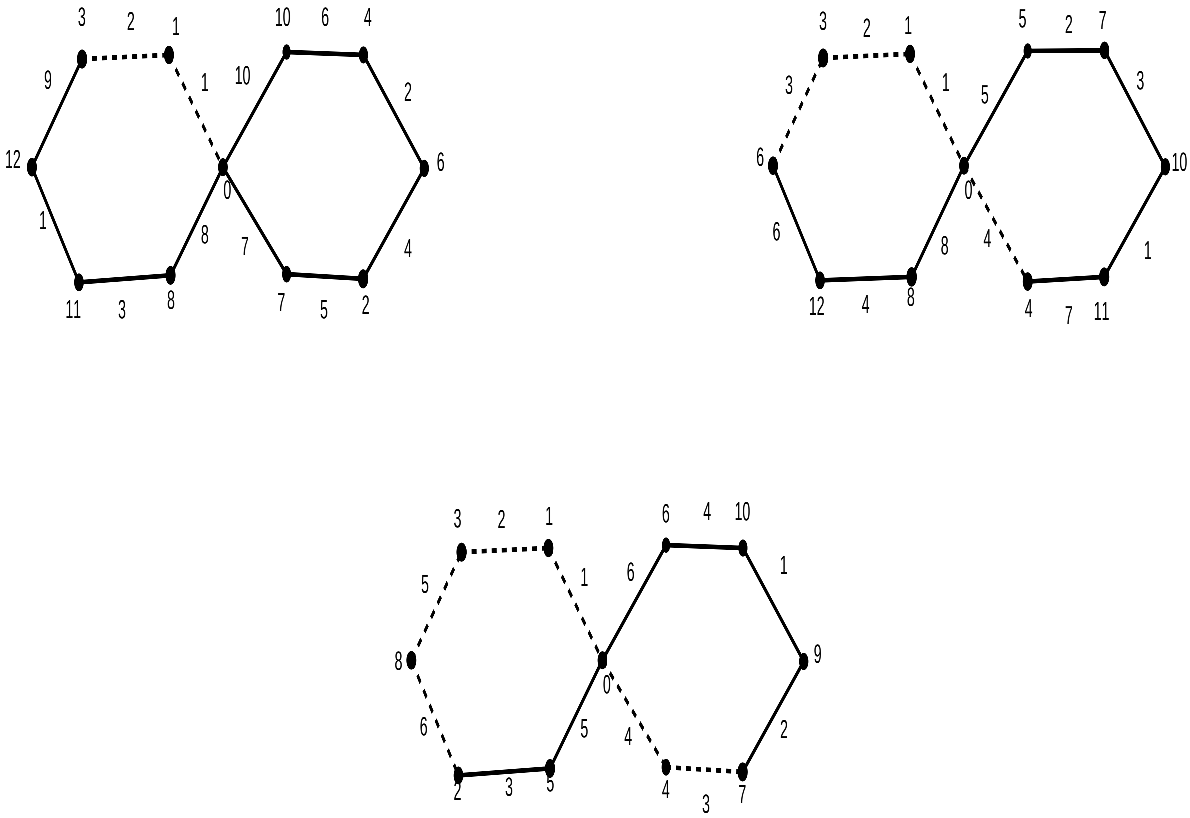


Figure 13.Graceful labelling on signed structure of T_6

4 Conclusion

The concept introduced in this paper gives room for numerous research problems.Particular graph arising in practical situations may be concentrated upon in this light.

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