TEJAS Thiagarajar College Journal **ISSN (Online):2456-4044**
June 2017, Vol 2(2), PP 82-92

A STUDY ON JOHNSON GRAPHS

S.M.Meena Rani [1] A. Balameena [2] S. Bharadha Devi [3]

[1] Associate Professor,
Department of Mathematics
V.V.Vanniaperumal College for Women
Virudhunagar

[2,3] M.Phil Scholars,
Department of Mathematics
V.V.Vanniaperumal College for Women
Virudhunagar

Abstract

The energy of a graph is defined as the sum of the absolute values of its eigenvalues. A strongly regular graph has the property that the number of common neighbours of two distinct vertices depends only on whether they are adjacent or non adjacent. The eigenvalues of a strongly regular graphs are determined by its parameters. In this paper an attempt has been made to identify classes of strongly regular Johnson graphs. Energy of such classes are determined by their parameters. For some other classes of Johnson graphs, Frame's method has been used or determining their energies.

key words: Johnson graph, Strongly regular graph, Eigenvalues, Energy.

2010 Mathematics Subject Classification: 05C50, 05C75

1 Introduction

Johnson graphs form a special class of undirected graphs defined from systems of sets. A regular graph is a graph where each vertex has the same number of neighbours. A strongly regular graph has the property that the number of common neighbours of two distinct vertices depends only on whether they are adjacent or nonadjacent vertices with parameters srg(n,k,a,c)and its complement is also strongly regular. The eigenvalues of a strongly regular graph are determined by its parameters. The energy of a graph is defined as the sum of the absolute values of its eigenvalues. In this paper an attempt has been made to identity classes of strongly regular Johnson graphs. Energy of such classes are determined by their parameters. For some other classes of Johnson graphs, Frame's method has been used for determining their energies.

2 Preliminaries

Definition 2.1 *Let G=(V,E) be a (p,q) graph. Let* $V=\{v_1,v_2,...v_p\}$. *The* $p\times p$ *matrix* $A=(a_{ij})$ *where* $a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{Otherwise} \end{cases}$ *is called the adjacency matrix of the graph G.*

Definition 2.2 *[1]:- Sum of the absolute values of the eigenvalues of the adjacency matrix of G is called energy of the graph G.*

Frame's method [2]:-

Frame's method is an alternative method for determining the characteristic polynomial of any matrix. Let A be a matrix of order n. Frame wrote down the characteristic equation corresponding to A in the following form:

$$|\lambda I - A| = \lambda^n - C_1\lambda^{n-1} - C_2\lambda^{n-2}......C_{n-1}\lambda - C_n = 0$$

The problem is to obtain the integral coefficients $C_1, C_2, ...$in the characteristic equation. For this we use the polynomial algorithm. This polynomial algorithm is described below.

Let A be the adjacency matrix of a graph. Construct the matrix $B_k^{\prime s}$ as follows :

$$B_1 = A(A - C_1 I), C_1 = trA,$$
$$C_2 = \tfrac{1}{2} trB_1,$$
$$B_1 = A(B_1 - C_2 I),$$
$$B_2 = A(B_1 - C_2 I),$$
..
..
..
$$B_{n-1} = A(B_{n-1} - C_{n-1} I),$$
$$C_n = \frac{trB_{n-1}}{n}.$$

Thus the coefficients $C_k^{\prime s}$ are generated recursively as traces of matrices.

Definition 2.3 *[3]:- A regular graph is a graph where each vertex has the same number of neighbours.*

(i.e) every vertex has the same degree or valency.

Definition 2.4 *[3]:- Let X be a regular graph that is neither complete nor empty. Then, X is said to be strongly regular with parameters (n,k,a,c)* *if it is k-regular, every pair of adjacent vertices has 'a' common neighbours, and every pair of distinct non adjacent vertices has 'c' common neighbours.*

If X is strongly regular with parameters (n,k,a,c),then it's complement $\bar{x}$ is also strongly regular with parameters $(n,\bar{k},\bar{a},\bar{c})$, where $\bar{k}$=n-k-1,
$\bar{a}$=n-2 -2k+c, $\bar{c}$=n-2k+a

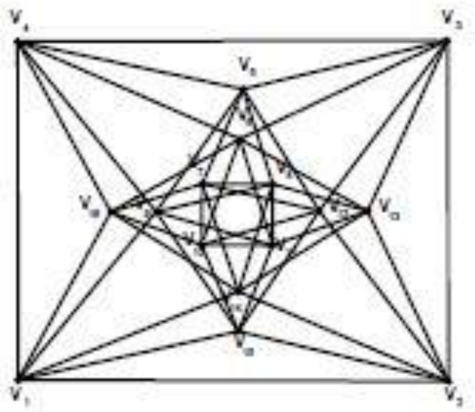

Fig.1 Shrikhande graph

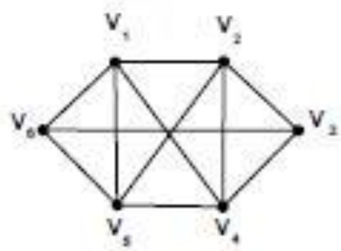

Fig.2

(i) Fig.1 graph is a shrikhande graph which is strongly regular with parameters (16,6,2,2)

(ii) Fig.2 graph is not strongly regular. since, there is any two adjacent vertices which is not contain same common neighbours.

Eigenvalues:-

Suppose A is the adjacency matrix of the (n,k,a,c) strongly regular graph X.We can determine the eigenvalues of the matrix A from the parameters of X.

The eigenvalues k,θ and τ are given by

$\theta = \frac{(a-c)+\sqrt{\Delta}}{2}, \tau = \frac{(a-c)-\sqrt{\Delta}}{2}$, k= valency of the graph
where $\Delta = (a-c)^2 - 4(k-c)$.

The multiplicities of the eigenvalues θ , τ are given by

$$m_\theta = \frac{1}{2}\left((n-1) - \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right) \text{ and}$$

$$m_\tau = \frac{1}{2}\left((n-1) + \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right)$$

Definition 2.5 *[3]:- Let γ,k,i be fixed positive integers $\gamma \geq k \geq i$. Let Ω be a fixed set of size γ. Define $J(\gamma, k, i)$ as follows: The vertices of $J(\gamma, k, i)$ are the subsets of Ω with size k. Join two vertices if their intersection has size i.*

Example 2.6 *The graph J(4,2,1) is given below in Fig.3*

Here γ=4, k=2, i=1, Let Ω={a,b,c,d}

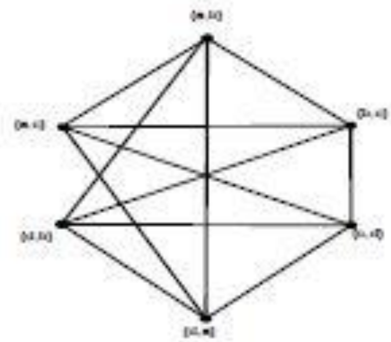

Fig 3. J(4,2,1)

Note 2.7 *Some of the Johnson graphs are strongly regular and some are not strongly regular.*

Example 2.8 *J(5,2,1) is a strongly regular graph*

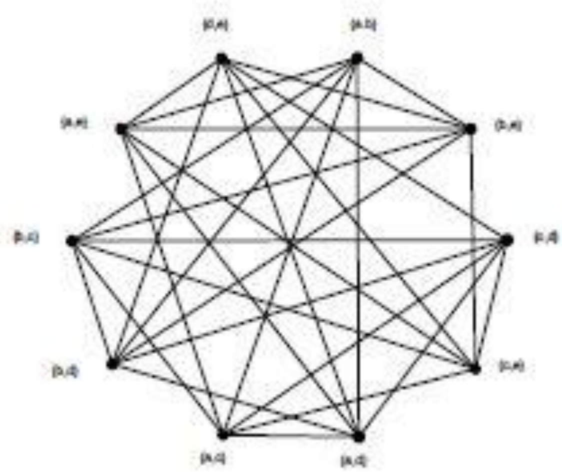

Fig 4.J(5,2,1)

Here γ=5, k=2, i=1, $\Omega=\{a, b, c, d, e\}$, n=10
In this graph, the number of vertices = $\binom{5}{2}$=10

$\therefore$ valency of the graph $= \binom{2}{1}\binom{5-2}{2-1}$=6

$\therefore$ number of common neighbours of any two adjacent vertices is 3.

Hence number of common neighbours of any two non adjacent vertices is 4. Parameters n,k,a,c are given below:

n	k	a	c
10	6	3	4

Example 2.9 *J(6,3,2) is not a strongly regular graph.*

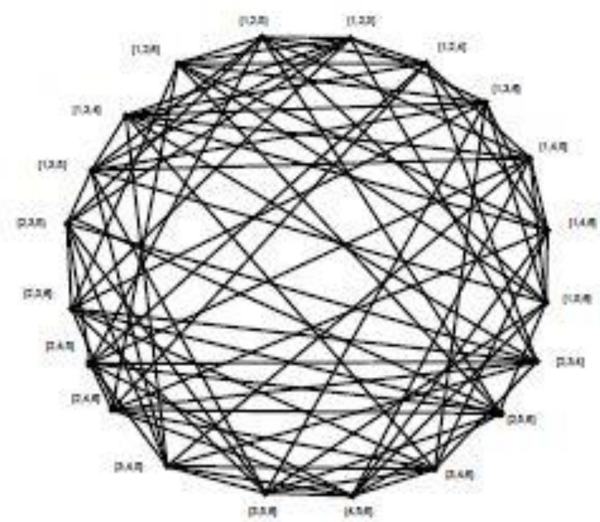

Fig 5.J(6,3,2)

For the nonadjacent vertices $\{a, b, d\}$, $\{a, c, f\}$, there is one common element a.

Then the common neighbours of $\{a, b, d\}$,$\{a, c, f\}$ are $\{a, b, c\}$, $\{a, b, f\}$, $\{a, c, d\}$,$\{a, d, f\}$.

For the vertices $\{a, b, c\}$,$\{d, e, f\}$, there is no common element.

Hence they are nonadjacent. These two vertices have no common neighbour.

$\therefore$ J(6,3,2) is not a strongly regular graph.

3 Strongly regular Johnson graphs

Theorem 3.1 *$J(\gamma,3,2)$ is not strongly regular for $\gamma>5$*

proof:

$J(\gamma,3,2)$ is a graph with $\binom{\gamma}{3}$ vertices.

The vertices of $J(\gamma,3,2)$ are the subsets of Ω with size 3 where Ω is a fixed set of size γ. Two vertices are adjacent if thier intersection has size 2.

Let $V(J)=\{V_1, V_2 \ldots V_l\}$

Consider two nonadjacent vertices V_i, V_j of J.

Then either $V_i \cap V_j = \Phi$ or $|V_i \cap V_j|=1$.

case(i):-

$V_i \cap V_j = \Phi$

Let $V_i = \{a,b,c\}$, $V_j = \{d,e,f\}$, such a pair exists since $\gamma \geq 6$

There is no common neighbour for V_i, V_j.

case(ii):-

$|V_i \cap V_i|=1$

Let $V_i = \{a,b,c\}, V_j = \{a,d,e\}$. Then {a,b,d} is a common neighbour of V_i and V_j.

$\therefore$ number of common neighbours of $V_i, V_j \geq 1$

Since $\gamma>5$, case(i) and case(ii) occur.

Hence $J(\gamma,3,2)$ is not a strongly regular graph when $\gamma > 5$.

Note 3.2 *$J(5,3,2)$ is strongly regular*

solution:-

Consider the graph J(5,3,2)

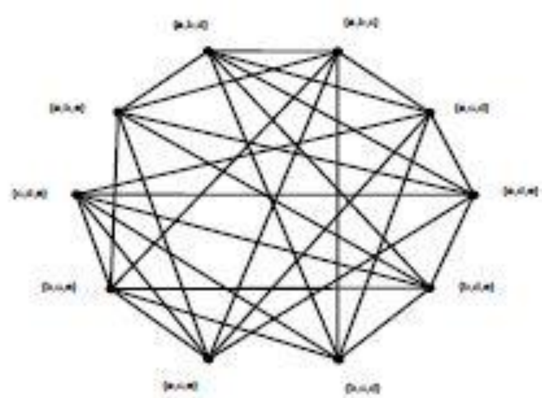

Fig 6.J(5,3,2)

Here γ=5, k=3, i=2, $\Omega=\{a,b,c,d,e\}$, n=10

In this graph, the number of vertices $=\binom{5}{3}$=10

$\therefore$ valency of the graph = 6

Consider two adjacent vertices of J(5,3,2)

Without loss of generality, let them be (a,b,c),(a,c,d)

The common neighbours of (a,b,c),(a,c,d) are (a,c,e),(b,c,d),(a,b,d)

ie., the vertices having a,c together with the members of $\Omega\setminus\{a,b,c,d\}$ and the vertex containing the distinct elements of chosen vertices.

$\therefore$ number of common neighbours of any two adjacent vertices is 3.

consider two nonadjacent vertices, say(a,c,e),(b,c,d)

The common neighbours are (a,c,d),(a,b,c),(c,d,e),(b,c,e)

Hence number of common neighbours of any two nonadjacent vertices is 4. Parameters n,k,a,c and the eigenvalues θ, τ together with their mulitiplicities are given below:

n	k	a	c	θ	τ	m_θ	m_τ
10	6	3	4	1	-2	4	5

$$\text{Energy of J(5,3,2)} = k+m_\theta|\theta| + m_\tau|\tau| = 20$$

Note 3.3 *J(4,3,2) is a complete graph*

Theorem 3.4 *Energy of $J(\gamma,2,1)$ is*

$$\frac{2\gamma^3-10\gamma^2+12\gamma}{\gamma-2}$$

proof:-

Consider the graph J(γ, 2, 1)

By construction the vertex set of J(γ, 2, 1) is the set of all subsets of size 2 of Ω Join two vertices if their intersection has size 1.

Let V(J)=$\{V_1, V_2 ... V_s\}$ in Ω

$\therefore$ the number of vertices of J(γ, 2, 1) is $\binom{\gamma}{2}$

$\therefore$ valency of the graph=2($\gamma-2$)

Consider two adjacent vertices of J(γ, 2, 1)

Without loss of generality, let them be $V_i=\{1,2\}, V_j=\{1,3\}$.

The common neighbours of V_i, V_j are $\{1,x\}$,

$x\in\Omega\setminus\{1,2,3\}$ and $\{2,3\}$.

$\therefore$ total number of common neighbours of any two adjacent vertices is $\gamma - 2$.

Any two nonadjacent vertices in J$(\gamma, 2, 1)$ should be disjoint 2-sets.

Without loss of generality let $V_i = \{1, 2\}$ and $V_j = \{3, 4\}$.

Any common neighbour of V_i, V_j should have one element from V_i and one element from V_j

Hence number of common neighbours is $2\times2=4$

$\therefore$ J$(\gamma, 2, 1)$ is a strongly regular graph with parameters

$(\binom{\gamma}{2}), 2(\gamma-2), \gamma-2, 4)$.

The eigenvalues θ,and τ are given by

$$\theta = \frac{a-c+\sqrt{\Delta}}{2}, \text{ where } \quad \Delta = (a\text{-}c)^2 + 4(a-c)$$
$$=(\gamma-2-4)^2 + 4(2\gamma-4-4)$$
$$=(\gamma-6)^2 + 4(2\gamma-8)$$
$$=\gamma^2 - 12\gamma + 36 + 8\gamma - 32$$
$$= \gamma^2 - 4\gamma + 4$$
$$=(\gamma-2)(\gamma-2) = (\gamma-2)^2$$
$$\sqrt{\Delta} = \gamma - 2$$
$$\theta = \frac{(\gamma-2-4)+\gamma-2}{2} = \frac{2\gamma-8}{2}$$
$$\tau = \frac{(\gamma-2-4)-\gamma+2}{2} = -2$$

$\therefore$ the eigenvalues θ and τ are $\theta = \frac{2\gamma-8}{2}, \tau =-2$

The multiplicities of the eigenvalues θ, τ are given by

$$m_\theta = \frac{1}{2}\left((n-1) - \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right)$$
$$=\frac{1}{2}\left(\frac{(n-1)(\gamma-2)-[3\gamma+\gamma n-6n-2]}{\gamma-2}\right)$$
$$=\frac{1}{2}\left(\frac{2(-\gamma+n+1)}{\gamma-2}\right)$$

$$n = \binom{\gamma}{2} = \frac{\gamma(\gamma-1)}{2} ==>$$
$$m_\theta = \frac{\gamma^2-3\gamma+2}{\gamma-2}$$
$$m_\tau = \frac{1}{2}\left((n-1) + \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right)$$
$$=\frac{1}{2}\left(\frac{(n-1)(\gamma-2)+3\gamma+\gamma n-6n-2]}{\gamma-2}\right)$$
$$=\frac{1}{2}\left(\frac{\gamma n-4n+\gamma}{\gamma-2}\right)$$

$$n = \binom{\gamma}{2} = \frac{\gamma(\gamma-1)}{2} ==>$$
$$m_\tau = \frac{\gamma^3-5\gamma^2+6\gamma}{2(\gamma-2)}$$

Energy of $J(\gamma,2,1) = k + m_\theta |\theta| + m_\tau |\tau|$

$$E(J(\gamma,2,1)) = 2(\gamma-2) + \frac{\gamma^2-3\gamma+2}{\gamma-2}\left|\frac{2\gamma-8}{2}\right| + \left(\frac{\gamma^3-5\gamma^2+6\gamma}{2(\gamma-2)}\right)|-2|$$

$$= \frac{4(\gamma-2)^2+(\gamma^2-3\gamma+2)(2\gamma-8)+2\gamma^3-10\gamma^2+12\gamma}{2(\gamma-2)}$$

$$= \frac{4\gamma^3-20\gamma^2+24\gamma}{2(\gamma-2)}$$

$$\therefore E(J(\gamma,2,1)) = \frac{2\gamma^3-10\gamma^2+12\gamma}{\gamma-2}$$

Energy of some strongly regular Johnson graphs otherthan $J(\gamma,2,1)$ are tabulated below:

Graph	n	k	a	c	θ	τ	m_θ	m_τ	Energy
J(5,3,2)	10	6	3	4	1	-2	4	5	20
J(6,4,3)	15	8	4	4	2	-2	5	9	36
J(7,4,3)	35	12	5	4	$1+\sqrt{33}/2$	$1-\sqrt{33}/2$	12	22	104.67
J(7,5,4)	21	10	5	4	3	-2	6	14	56
J(8,5,4)	56	15	6	4	$1+2\sqrt{3}$	$1-2\sqrt{3}$	17	38	184.53
J(8,6,5)	28	12	6	4	4	-2	7	20	80
J(9,7,6)	36	14	5	4	-2	-2	8	27	108
J(10,8,7)	45	16	8	4	6	-2	9	35	140

Eigenvalues of some Johnson graphs which are not strongly regular by using Frame's method:

Graph	Characteristic polynomial	Energy
J(6,3,2)	$\lambda^{20} - 89\lambda^{18} - 240\lambda^{17} + 2124\lambda^{16} - 3555\lambda^{15} - 4041\lambda^{14}$ $+3918\lambda^{13} + 3535\lambda^{12} - 342\lambda^{11} + 1826\lambda^{10} + 1252\lambda^{9}$ $-1280\lambda^{8} - 563\lambda^{7} - 173\lambda^{6} - 2043\lambda^{5} - 412\lambda^{4}$ $-383\lambda^{3} - 412\lambda^{2} - 19\lambda + 1622$	57.162
J(7,3,2)	$\lambda^{35} - 207\lambda^{33} - 697\lambda^{32} - 1315\lambda^{31} - 163\lambda^{30} - 2333\lambda^{29}$ $-4481\lambda^{28} + 2907\lambda^{27} + 3640\lambda^{26} + 595\lambda^{25} + 225\lambda^{24} + 63\lambda^{23} + 1831\lambda^{22}$ $-1790\lambda^{21} - 1991\lambda^{20} - 412\lambda^{4} - 383\lambda^{3} - 412\lambda^{2} - 1357\lambda^{19} - 559\lambda^{18}$ $+1167\lambda^{17} - 284\lambda^{16} - 1144\lambda^{15} + 771\lambda^{14} + 313\lambda^{13} + 754\lambda^{12} - 558\lambda^{11}$ $+15\lambda^{10} + 749\lambda^{9} + 607\lambda^{8} - 1023\lambda^{7} + 915\lambda^{6} - 179\lambda^{5} + 108\lambda^{4}$ $-737\lambda^{3} + 380\lambda^{2} + 197\lambda + 625$	283.438

4 conclusion

In this paper, energy has been determined for some strongly regular johnson graph parameter and this type of work may be extended to other Johnson graphs as well as to some other families of graphs.

References

[1] I. Gutman, The energy of a graph, *Ber. Math.-Stat. Sekt. Forschungszentrum Graz* 103(1978), 1-22.

[2] K.Balasubramanian,The Use of frame's method for the Characteristic polynomials of Chemical graphs, *Theoret. Chim.acta(berl.)*, 1984,65, pp:49-58

[3] Chris Godsil and Gordon Royle Algebraic Graph theory, *Springer-Verlog*, 2001, 3,pp:217-221, 226-230.